# Quantum Field Theory

## Set 4: solutions

## Appetizer

We will use

$$A_\mu(x) = \int d\Omega_{\vec{k}} (a_\mu(\vec{k}) e^{-ik\cdot x} + a^\dagger_\mu(\vec{k}) e^{ik\cdot x}) \tag{1}$$

where by $x$ and $k$ we denote the 4-vectors, i.e. $k \cdot x = k_\mu x^\mu = \omega t - \vec{k}\cdot\vec{x}$. This implies

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - (\mu \leftrightarrow \nu) = \int d\Omega_{\vec{k}} (-ik_\mu a_\nu(\vec{k}) e^{-ik\cdot x} + ik_\mu a^\dagger_\nu(\vec{k}) e^{ik\cdot x}) - (\mu \leftrightarrow \nu) \tag{2}$$

Using

$$\langle 0|a_\mu(\vec{k}_1) a^\dagger_\nu(\vec{k}_2)|0\rangle = -\eta_{\mu\nu}(2\pi)^3 2\omega_{\vec{k}} \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \tag{3}$$

we thus obtain

$$\langle 0|A_\mu(x)|\epsilon(\vec{k})\rangle = -\epsilon_\mu(\vec{k}) e^{-ik\cdot x} \qquad \langle 0|F_{\mu\nu}(x)|\epsilon(\vec{k})\rangle = i(k_\mu \epsilon_\nu(\vec{k}) - k_\nu \epsilon_\mu(\vec{k})) e^{-ik\cdot x} \tag{4}$$

These matrix elements describe how the quantum field $A_\mu$ is affected by the interacting with a photon. Seen the other way around, it describes how $A_\mu(x)|0\rangle$ creates a linear combination of one-photon states.

## Exercise 1

The polarization of a photon of momentum $k_\mu$ is defined by the constraint:

$$\varepsilon_\mu k^\mu = 0.$$

Let's define a four-vector $\bar{k}^\mu$ with components $\bar{k}^0 = k^0$ and $\bar{k}^i = -k^i$. Note that $k^\mu$ and $\bar{k}^\mu$ form a complete basis of the longitudinal-time subspace. In terms of $k^\mu$, $\bar{k}^\mu$ and $\varepsilon^\mu$, the transverse polarization vector is written as:

$$\varepsilon^\perp_\mu = \left( g_{\mu\nu} - \frac{k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu}{k \cdot \bar{k}} \right) \varepsilon^\nu = \varepsilon_\mu - \left( \frac{\bar{k}\cdot\varepsilon}{\bar{k}\cdot k} \right) k_\mu.$$

Note that $\varepsilon^\perp_\mu$ satisfies $\varepsilon^\perp_\mu k^\mu = \varepsilon^\perp_\mu \bar{k}^\mu = 0$, and that these conditions are of course Lorentz-invariant because written in terms of dot products[1]. Moreover, writing more explicitly $k^\mu = k^0(1, \vec{n})$ and $\bar{k}^\mu = k^0(1, -\vec{n})$ it is easy to prove that the condition $\varepsilon_\mu k^\mu = 0$ implies $\varepsilon_0 = \vec{\varepsilon}\cdot\vec{n}$ and in turn that:

$$\begin{aligned} \varepsilon^\perp_0 &= 0, \\ \frac{\bar{k}\cdot\varepsilon}{\bar{k}\cdot k} &= \frac{\varepsilon^0}{k^0}. \end{aligned}$$

Finally, $\varepsilon^\perp_\mu k^\mu = \varepsilon^\perp_\mu \bar{k}^\mu = 0$ implies $\varepsilon^\perp_i k^i = 0$.
Now consider a generic Lorentz transformation acting on $\varepsilon^\perp_\mu$ and transforming it into ${\varepsilon'}^\perp_\mu = {\varepsilon'}_\mu - \frac{\varepsilon^0}{k^0} {k'}_\mu$. We get:

$$\begin{aligned} {\varepsilon'}^\perp_0 &= {\varepsilon'}_0 - \frac{\varepsilon^0}{k^0} {k'}_0 && \neq 0, \\ {\varepsilon'}^\perp_i {k'}^i &= -{\varepsilon'}^\perp_0 {k'}^0 && \neq 0. \end{aligned}$$

Besides rotations, there is only one case in which the equations above are not verified (i.e. in which ${\varepsilon'}^\perp_0 = 0$, and consequently ${\varepsilon'}^\perp_i {k'}^i = 0$), namely the case of a longitudinal boost: such a boost leaves the transverse components

[1] Notice however that the functional form of $\bar{k}_\mu$ in terms of the components of $k_\mu$ is not preserved by Lorentz transformations.

of any fourvector untouched and mixes the time and longitudinal components, which for ${\varepsilon'}^{\perp}_{\mu}$ are both 0.
For generic transformation one can define:

$$\tilde{\varepsilon}^{\perp}_{\mu} \equiv {\varepsilon'}^{\perp}_{\mu} + \left(\frac{\varepsilon^0}{k^0} - \frac{{\varepsilon'}^0}{{k'}^0}\right) k'_{\mu},$$

finding:

$$\begin{aligned} \tilde{\varepsilon}^{\perp}_{0} &= {\varepsilon'}_0 - \frac{{\varepsilon'}^0}{{k'}^0}{k'}_0 &= 0, \\ \tilde{\varepsilon}^{\perp}_{i}{k'}^i &= -\tilde{\varepsilon}^{\perp}_{0}{k'}^0 &= 0. \end{aligned}$$

Note that in the special case of longitudinal boost one has $\tilde{\varepsilon}^{\perp}_{\mu} = {\varepsilon'}^{\perp}_{\mu}$, as can be seen from the definition of $\varepsilon^{\perp}_{\mu}$ replacing all fourvectors by their primed counterparts. In general, however, Lorentz transforming $\frac{\bar{k}\cdot\varepsilon}{\bar{k}\cdot k} = \frac{\varepsilon^0}{k^0}$ you get $\frac{\bar{k}'\cdot\varepsilon'}{\bar{k}'\cdot k'} \neq \frac{{\varepsilon'}^0}{{k'}^0}$ because $\bar{k}'$ cannot be written as $(k'_0, -\vec{k}')$.
One important point to notice is the following. Since the condition of orthogonality ($\tilde{\varepsilon}^{\perp}_{i}{k'}^i = 0$) and of null time component ($\tilde{\varepsilon}^{\perp}_{0} = 0$) are not Lorentz-invariant, if an observer defines a fourvector which just contains the two physical transverse photon polarizations, another generic observer will see that fourvector as containing three photon polarizations, meaning that the projection on the physical subspace is an observer-dependent statement. So when we define the vector $\varepsilon^{\perp}_{\mu}$, we are defining an object that transforms as a fourvector, but in a weak sense: it is true that $\Lambda : \varepsilon^{\perp}_{\mu} \to \Lambda_{\mu}{}^{\nu}\varepsilon^{\perp}_{\nu} \equiv {\varepsilon'}^{\perp}_{\mu}$, but ${\varepsilon'}^{\perp}_{\mu}$ does *not* share the basic, defining property of $\varepsilon^{\perp}_{\mu}$, namely the '$\perp$'. If instead we want a Lorentz-transformed vector which shares the same defining properties as $\varepsilon^{\perp}_{\mu}$ we have to implement a *nonlinear* transformation $\Lambda_{NL} : \varepsilon^{\perp}_{\mu} \to \Lambda_{\mu}{}^{\nu}\varepsilon^{\perp}_{\nu} - \frac{{\varepsilon'}^0}{{k'}^0}\Lambda_{\mu}{}^{\nu}k_{\nu} \equiv \tilde{\varepsilon}^{\perp}_{\mu}$.
At the end, as far as physical applications are concerned, these remarks, even though conceptually important, are quite harmless since we'll see that gauge invariance implies $M^{\mu}k_{\mu} = 0$, with $M^{\mu}$ a physical scattering amplitude, so that ${M'}^{\mu}{\varepsilon'}^{\perp}_{\mu} = {M'}^{\mu}\tilde{\varepsilon}^{\perp}_{\mu}$ for all observers, but it is important to keep in mind the distinction between ${\varepsilon'}^{\perp}_{\mu}$ and $\tilde{\varepsilon}^{\perp}_{\mu}$ in cases in which even the longitudinal part enters the game.

We now decompose the transverse polarization vector $\varepsilon_{\perp}$ into helicity eigenstates $\varepsilon_{\pm}$

$$\varepsilon_{\perp}(k) = c_{+}\varepsilon_{+}(k) + c_{-}\varepsilon_{-}(k) \tag{5}$$

where

$$\exp(-iJ\cdot\hat{\mathbf{n}}\,\phi)\varepsilon_{\pm}(k) = e^{\mp i\phi}\varepsilon_{\pm}(k) \tag{6}$$

for $\hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}|$. We want to understand how $\varepsilon_{\pm}$ behaves under Lorentz transformations. Let us start considering the reference frame where the photon has momentum $\tilde{k}^{\mu} = (\omega, 0, 0, \omega)$. Of course, this is not unique but it is identified only up to a transformation of the little group. In this frame, we can chose $\varepsilon^{\mu}_{+}(\tilde{k}) = \frac{1}{\sqrt{2}}(0, -1, -i, 0)$ and $\varepsilon^{\mu}_{-}(\tilde{k}) = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$. Under $W \in ISO(2)$ these vectors are not mapped into transverse vectors. To see this explicitely, let us recall that every transformation of this type can be rewritten as

$$W(\alpha, \beta, \phi) = S(\alpha, \beta)R(\phi) \tag{7}$$

where $R$ is a rotation around the third axis while $S$ is generated exponentiating $J^1 - K^2$ and $J^2 + K^1$ with paramters $\alpha$ and $\beta$. From the $(\frac{1}{2}, \frac{1}{2})$ representation of the generators it is not difficult to compute $S(\alpha, \beta)$. The result is

$$S(\alpha, \beta) = \begin{pmatrix} \frac{1}{2}(\alpha^2+\beta^2) & -\beta & \alpha & -\frac{1}{2}(\alpha^2+\beta^2) \\ -\beta & 1 & 0 & \beta \\ \alpha & 0 & 1 & -\alpha \\ \frac{1}{2}(\alpha^2+\beta^2) & -\beta & \alpha & -\frac{1}{2}(\alpha^2+\beta^2) \end{pmatrix} \tag{8}$$

Therefore, under a little group transformation, up to a phase given by the rotation, we end up with a polarization vector shifted along the longitudinal direction

$$W(\alpha, \beta, \phi)^{\mu}{}_{\nu}\,\varepsilon^{\nu}_{\pm}(\tilde{k}) = e^{\mp i\phi}(\varepsilon^{\mu}_{\pm}(\tilde{k}) + \frac{\mp\alpha - i\beta}{\sqrt{2}\omega}\tilde{k}^{\mu}) \tag{9}$$

Notice however, that $\varepsilon^\mu_\pm(\tilde{k}) + c\tilde{k}^\mu$ is equivalent to $\varepsilon^\mu_\pm(\tilde{k})$. This result shows that the (Coulomb) transversality condition is not covariant. The general case is not different. At first, let us define the standard helicity basis Lorentz transformation $\Lambda_k$ which maps $\tilde{k}$ into $k$ and define $\varepsilon^\mu_\pm(k) = \Lambda^\mu_{k\,\nu}\,\varepsilon^\nu_\pm(\tilde{k})$. At this point we see that

$$\Lambda^\mu{}_\nu\,\varepsilon^\nu_\pm(k) = (\Lambda_{\Lambda k})^\mu{}_\rho(\Lambda^{-1}_{\Lambda k}\Lambda\Lambda_k)^\rho{}_\nu\,\varepsilon^\nu_\pm(\tilde{k}) \tag{10}$$

The second transformation belongs to the little group of $\tilde{k}$. Therefore, from the previous comments we find

$$\begin{aligned}(\Lambda^{-1}_{\Lambda k}\Lambda\Lambda_k)^\mu{}_\nu\,\varepsilon^\nu_\pm(\tilde{k}) &= S(\alpha(\Lambda,\tilde{k}),\beta(\Lambda,\tilde{k}))^\mu{}_\rho R(\phi((\Lambda,\tilde{k})))^\rho{}_\nu\varepsilon^\nu_\pm(\tilde{k}) \\ &= e^{\mp i\phi(\Lambda,\tilde{k})}(\varepsilon^\mu_\pm(\tilde{k}) + \frac{\mp\alpha(\Lambda,\tilde{k}) - i\beta(\Lambda,\tilde{k})}{\sqrt{2}\omega}\tilde{k}^\mu)\end{aligned} \tag{11}$$

and thus

$$\Lambda^\mu{}_\nu\,\varepsilon^\nu_\pm(k) = e^{\mp i\phi(\Lambda,\bar{k})}(\varepsilon^\mu_\pm(\Lambda k) + \frac{\mp\alpha(\Lambda,\bar{k}) - i\beta(\Lambda,\bar{k})}{\sqrt{2}\omega}(\Lambda k)^\mu) \tag{12}$$

Again, in the right hand side we ended up with a shifted vector which is equivalent to $\varepsilon^\mu_\pm(\Lambda k)$.

## Exercise 2

Let us consider the Lagrangian of a massive vector field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2A_\mu A^\mu,$$

from which we can compute the conjugate momentum of the field $A_\mu$:

$$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu}.$$

$\Pi^0$ is vanishing, which is a consequence of the fact that the field $A_0$ is not a dynamical variable. The only non trivial quantities are then:

$$\Pi^i = -\partial^0A^i + \partial^iA^0.$$

The equations of motion following from the above Lagrangian can be divided into a set of three dynamical equations:

$$0 = \partial_\mu F^{\mu j} + M^2A^j = \partial_0F^{0j} + \partial_iF^{ij} + M^2A^j = \ddot{A}^j - \partial^j\dot{A}_0 + \partial_iF^{ij} + M^2A^j,$$

and constraint:

$$0 = \partial_\mu F^{\mu 0} + M^2A^0 \implies A_0 = -\frac{1}{M^2}\partial_i\Pi^i,$$

which lets us express the non-dynamical variable as a function of the momenta. The Hamiltonian reads:

$$H = \int d^3x\left(\Pi^\mu\dot{A}_\mu - \mathcal{L}\right) = \int d^3x\left(\Pi^i\dot{A}_i - \mathcal{L}\right) = \int d^3x\left(-\Pi^i\Pi_i + \Pi^i\partial_iA_0 - \mathcal{L}\right),$$

where in the last step we have used the definition of $\Pi^i$ in terms of $\dot{A}^i$. Expanding and eliminating $A_0$, we get

$$\begin{aligned}H &= \int d^3x\left(-\Pi^i\Pi_i - \frac{1}{M^2}\Pi^i\partial_i(\partial_j\Pi^j) - \mathcal{L}\right) = \int d^3x\left(-\Pi^i\Pi_i - \frac{1}{M^2}\Pi^i\partial_i(\partial_j\Pi^j) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M^2A_\mu A^\mu\right) \\ &= \int d^3x\left(\Pi^i\Pi^i + \frac{1}{M^2}(\partial_i\Pi^i)(\partial_j\Pi^j) + \frac{1}{2}F_{0j}F^{0j} + \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}M^2A_0^2 - \frac{1}{2}M^2A_iA^i\right) \\ &= \int d^3x\,\frac{1}{2}\left(\Pi^i\Pi^i + \frac{1}{M^2}(\partial_i\Pi^i)^2 + \frac{1}{2}F^{ij}F^{ij} + M^2A^iA^i\right).\end{aligned}$$

In order to quantize this theory we need to impose a set of canonical commutation relations. Here a subtlety arises since, because of the constraint relating $A_0$ and the $\Pi^i$, we cannot impose a vanishing commutation relation between all the $A_\mu$. A correct set of commutation relation is instead:

$$\begin{aligned}&\left[A_i(\vec{x},t),\,\Pi^j(\vec{y},t)\right] = i\delta^j_i\delta^3(\vec{x}-\vec{y}), \\ &[A_i(\vec{x},t),\,A_j(\vec{y},t)] = \left[\Pi^i(\vec{x},t),\,\Pi^j(\vec{y},t)\right] = \left[A_0(\vec{x},t),\,\Pi^j(\vec{y},t)\right] = 0, \\ &[A_i(\vec{x},t),\,A_0(\vec{y},t)] = \left[A_i(\vec{x},t),\,-\frac{1}{M^2}\partial_m\Pi^m(\vec{y},t)\right] = \frac{i}{M^2}\partial^{(x)}_i\delta^3(\vec{x}-\vec{y}).\end{aligned}$$

We can check the consistency of the above commutation relations considering the commutator of the field and the momenta with the Hamiltonian. Notice that the commutation relations are defined at equal time but since $H$ is independent of time we can compute it at any time:

$$\begin{aligned}\left[H,\,\Pi^j(\vec{x},t)\right] &\equiv -i\dot{\Pi}^j(\vec{x},t) = \int d^3y\left(\frac{1}{2}F^{mn}\left[F_{mn}(\vec{y},t),\,\Pi^j(\vec{x},t)\right] - M^2A^m\left[A_m(\vec{y},t),\,\Pi^j(\vec{x},t)\right]\right)\\ &= \int d^3y\left(\,F^{mn}\left[\partial_m^{(y)}A_n(\vec{y},t),\,\Pi^j(\vec{x},t)\right] - M^2A^m\left[A_m(\vec{y},t),\,\Pi^j(\vec{x},t)\right]\right)\\ &= i\int d^3y\left(\,F^{mj}\partial_m^{(y)}\delta^3(\vec{x}-\vec{y}) - M^2A^j\delta^3(\vec{x}-\vec{y})\right) = -i\partial_m\,F^{mj}(\vec{x},t) - iM^2A^j(\vec{x},t),\end{aligned}$$

$$\begin{aligned}\left[H,\,A_j(\vec{x},t)\right] &\equiv -i\dot{A}_j(\vec{x},t) = \int d^3y\left(-\Pi_i\left[\Pi^i(\vec{y},t),\,A_j(\vec{x},t)\right] + \frac{1}{M^2}(\partial_m^{(y)}\Pi^m)\left[(\partial_n^{(y)}\Pi^n),\,A_j(\vec{x},t)\right]\right)\\ &= \int d^3y\left(i\Pi_j(\vec{y},t)\delta^3(\vec{x}-\vec{y}) + i\frac{1}{M^2}(\partial_m\Pi^m)\partial_j^{(x)}\delta^3(\vec{x}-\vec{y})\right) = i\Pi_j(\vec{x},t) + \frac{i}{M^2}\partial_j(\partial_m\Pi^m)(\vec{x},t).\end{aligned}$$

Finally taking the time derivative of the second equation and using the first we get:

$$\begin{aligned}&\ddot{A}^j = -\dot{\Pi}^j - \frac{1}{M^2}\partial_0\partial^j(\partial_m\Pi^m) = -\dot{\Pi}^j + \partial_0\partial^jA_0 = -\partial_m\,F^{mj} - M^2A^j + \partial_0\partial^jA_0\\ \Longrightarrow\ &\partial_0\partial^0A^j - \partial_0\partial^jA^0 + \partial_m\,F^{mj} + M^2A^j = 0.\end{aligned}$$

## Homework

The energy momentum tensor is

$$T^{\mu\nu} = -\partial^\mu A^\rho\partial^\nu A_\rho + \frac{\eta^{\mu\nu}}{2}(\partial_\alpha A_\beta)(\partial^\alpha A^\beta).$$

Using that the Lorentz generator for spin 1 fields is

$$(\mathcal{J}^{\rho\sigma})_\nu^{\ \gamma} = -i(\delta_\nu^\rho\eta^{\sigma\gamma} - \delta_\nu^\sigma\eta^{\rho\gamma}),$$

we then find

$$\begin{aligned}J^{0ij} &= -x^i\partial^0A^\gamma\partial^jA_\gamma + x^j\partial^0A^\gamma\partial^iA_\gamma,\\ S^{0ij} &= -(\partial^0A^i)A^j + (\partial^0A^j)A^i.\end{aligned}$$

Let us first compute the explicit expressions in terms of ladder operators by using the expansion

$$A_\mu(x) = \int d\Omega_{\vec{k}}\left[a_\mu(\vec{k},t) + a_\mu^\dagger(-\vec{k},t)\right]e^{i\vec{k}\cdot\vec{x}}.$$

The first piece of $L^{ij}$ is

$$-\int d^3x\,x^i\dot{A}^\gamma\partial^jA_\gamma = \int d^3x(-x^i)\int d\Omega_{\vec{k}}(-ik_0)\left[a^\gamma(\vec{k},t) - a^{\gamma\,\dagger}(-\vec{k},t)\right]e^{i\vec{k}\cdot\vec{x}}\int d\Omega_{\vec{p}}(-ip^j)\left[a_\gamma(\vec{p},t) + a_\gamma^\dagger(-\vec{p},t)\right]e^{i\vec{p}\cdot\vec{x}}.$$

Using $x^ie^{i\vec{p}\cdot\vec{x}} = -i\frac{\partial}{\partial p^i}e^{i\vec{p}\cdot\vec{x}}$ and integrating by parts, we find

$$\iint d\Omega_{\vec{k}}d\Omega_{\vec{p}}\int d^3xe^{i(\vec{k}+\vec{p})\cdot\vec{x}}(-ik_0)\left[a^\gamma(\vec{k},t) - a^{\gamma\,\dagger}(-\vec{k},t)\right]\left(-i\frac{\partial}{\partial p^i}\right)\left[a_\gamma(\vec{p},t) + a_\gamma^\dagger(-\vec{p},t)\right](-ip^j).$$

The integral over $d^3x$ gives $(2\pi)^3\delta^2(\vec{p}+\vec{k})$, thus recalling the form of the measure $d\Omega_{\vec{k}} = \frac{d^3k}{2k_0(2\pi)^3}$, we arrive at

$$-\int d^3x\,x^i\dot{A}^\gamma\partial^jA_\gamma = \int d\Omega_{\vec{p}}\frac{i}{2}\left[a^\gamma(-\vec{p},t) - a^{\gamma\,\dagger}(\vec{p},t)\right]\frac{\partial}{\partial p^i}p^j\left[a_\gamma(\vec{p},t) + a_\gamma^\dagger(-\vec{p},t)\right].$$

Similarly

$$\int d^3x\, x^j \dot{A}^\gamma \partial^i A_\gamma = -\int d\Omega_{\vec{p}} \frac{i}{2} \left[a^\gamma(-\vec{p},t) - a^{\gamma\,\dagger}(\vec{p},t)\right] \frac{\partial}{\partial p^j} p^i \left[a_\gamma(\vec{p},t) + a^\dagger_\gamma(-\vec{p},t)\right].$$

Whence we get

$$J^{ij} = i\int d\Omega_{\vec{p}}\, a^{\gamma\,\dagger}(\vec{p}) \left[p^i \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p^i}\right] a_\gamma(\vec{p}).$$

The computation of the spin part is easier. With similar steps we find

$$-\int d^3x (\partial^0 A^i) A^j = \int d\Omega_{\vec{p}} \frac{i}{2} \left[a^i(\vec{p},t) - a^{i\,\dagger}(-\vec{p},t)\right] \left[a^j(-\vec{p},t) + a^{j\,\dagger}(\vec{p},t)\right],$$

$$\int d^3x (\partial^0 A^j) A^i = -\int d\Omega_{\vec{p}} \frac{i}{2} \left[a^j(\vec{p},t) - a^{j\,\dagger}(-\vec{p},t)\right] \left[a^i(-\vec{p},t) + a^{i\,\dagger}(\vec{p},t)\right].$$

Then we conclude

$$S^{ij} = -i\int d\Omega_{\vec{p}} \left[a^{i\,\dagger}(\vec{p}) a^j(\vec{p}) - a^{j\,\dagger}(\vec{p}) a^i(\vec{p})\right].$$

As in exercise 1, consider now $L(\vec{q}) = q_\rho a^\rho(\vec{q})$. The total angular momentum is a physical observable, hence we expect $[L, M^{ij}] \propto L$. However the same must not be necessarily true for the orbital and spin part alone. To see if this is the case let us compute the commutator of $L(\vec{q})$ with $J^{ij}$ and $S^{ij}$. Take $J^{ij}$ first:

$$\begin{aligned}
[L(\vec{q}), J^{ij}] &= i\int d\Omega_{\vec{p}}\, q_\rho \left[a^\rho(\vec{q}), a^{\gamma\,\dagger}(\vec{p})\right] \left[p^i \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p^i}\right] a_\gamma(\vec{p}) \\
&= i q^\gamma \left[q^i \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial q^i}\right] a_\gamma(\vec{q}) \\
&= i \left[q^i \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial q^i}\right] L(\vec{q}) - i\left[q^i a_j(\vec{q}) - q^j a_i(\vec{q})\right].
\end{aligned}$$

In passing from the first to the second line we used the commutation relation $[a_\mu(\vec{p}), a^\dagger_\nu(\vec{q})] = -\eta_{\mu\nu} 2q_0 (2\pi)^3 \delta^3(\vec{p}-\vec{k})$, while in the last we used[2] $\left[q^\gamma, \frac{\partial}{\partial q^j}\right] = -\delta^\gamma_j - \delta^\gamma_0 \frac{q^j}{q^0}$.
Let us now do the same computation for the spin:

$$\begin{aligned}
[L(\vec{q}), S^{ij}] &= -i\int d\Omega_{\vec{p}}\, q^\rho \left\{\left[a_\rho(\vec{q}), a^{i\,\dagger}(\vec{p})\right] a^j(\vec{p}) - \left[a_\rho(\vec{q}), a^{j\,\dagger}(\vec{p})\right] a^i(\vec{p})\right\} \\
&= i\left[q^j a^i(\vec{q}) - q^i a^j(\vec{q})\right].
\end{aligned}$$

Using $a^i(\vec{q}) = -a_i(\vec{q})$, we finally find

$$[L(\vec{q}), M^{ij}] = i\left[q^i \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial q^i}\right] L(\vec{q}).$$

Then, defining $L(q) = L(\vec{q}) e^{-iq^0 t}$, we get:

$$L(x) = \partial^\mu A^-_\mu(x) = \int d\Omega_{\vec{q}}\, e^{-iqx} L(\vec{q}) \implies [L(x), M^{ij}] = -i(x^i \partial_j - x^j \partial_i) L(x).$$

We thus conclude that $M^{ij}$ is a well defined observable, but nor $J^{ij}$ nor $S^{ij}$ are alone. In particular the spin of the photon is not a well defined observable. Notice however that we can define the following *helicity* operator:

$$h = -i\epsilon_{ijk} \int d\Omega_{\vec{p}} a^{i\,\dagger}(\vec{p}) a^j(\vec{p}) p^k.$$

For single particle states this measures the projection of $S^k = \frac{1}{2}\epsilon_{ijk} S^{ij}$ onto the direction of the momentum, i.e. $h = \vec{p}\cdot\vec{S}$. Then the same computation done before shows

$$[L(\vec{q}), h] = i\epsilon_{ijk} q^j a^i(\vec{q}) q^k = 0.$$

This hence shows that the operator $h$ is a well defined observable. In other words the spin of a photon cannot be measured, while the helicity can[3].

[2]This can be proved taking into account that $q$ is on-shell, so $\frac{\partial q^0}{\partial q^j} = \frac{q^j}{q^0}$.
[3]This result could have been expected; see Weinberg 2.5, *single particle states.*